

The \mathbf{s} -Eulerian polynomials have only real roots

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August 21, 2012

Abstract

In recent years there has been much interest in studying the roots of various generalized Eulerian polynomials. These generalizations arise from enumerative and algebraic generalizations of the descent generating polynomials of permutations. In this paper we present a novel approach.

We interpret Eulerian polynomials as the generating polynomials of a statistic over inversion sequences. Recently, Savage and Schuster considered inversion sequences (also known as Lehmer codes) and introduced, for any sequence \mathbf{s} of positive integers, a generalization, called \mathbf{s} -inversion sequences. They defined various statistics on \mathbf{s} -inversion sequences and explored their connection with lecture hall polytopes and with other combinatorial families.

Let $\mathbf{E}_n^{(\mathbf{s})}(x)$ denote the generating polynomial of the *ascent* statistic over all \mathbf{s} -inversion sequences of length n . The main result of this paper is that, for any sequence \mathbf{s} of positive integers, the \mathbf{s} -Eulerian polynomial, $\mathbf{E}_n^{(\mathbf{s})}(x)$, has only real roots. This result is first shown to generalize many existing results about the real-rootedness of various Eulerian polynomials. It is then extended to several q -analogs of $\mathbf{E}_n^{(\mathbf{s})}(x)$. We show for the first time that the MacMahon–Carlitz q -Eulerian polynomial has only real roots for positive real q . The same holds true for the (des, finv)-generating polynomials and the (des, fmaj)-generating polynomials for the hyperoctahedral group, B_n , and the wreath product groups $C_k \wr \mathfrak{S}_n$, confirming conjectures of Chow and Gessel, and Chow and Mansour.

We show that the new results have the following geometric consequence: for any sequence \mathbf{s} of positive integers, the \mathbf{h}^* -polynomial of the \mathbf{s} -lecture hall polytope has only real roots and therefore its coefficient sequence is unimodal and log-concave.

1 Introduction

For a sequence $\mathbf{s} = \{s_i\}_{i \geq 1}$ of positive integers, the n -dimensional \mathbf{s} -inversion sequences are defined by

$$\mathbf{I}_n^{(\mathbf{s})} = \{(e_1, \dots, e_n) \in \mathbb{Z}^n \mid 0 \leq e_i < s_i \text{ for } 1 \leq i \leq n\}.$$

When $\mathbf{s} = (1, 2, 3, \dots)$, there are well-known bijections between $\mathbf{I}_n^{(\mathbf{s})}$ and \mathfrak{S}_n , the set of permutations of $\{1, 2, \dots, n\}$. We use $\mathbf{I}_n^{(\mathbf{s})}$ to generalize results about the distribution of statistics on \mathfrak{S}_n .

The *ascent set* of an \mathbf{s} -inversion sequence $\mathbf{e} \in \mathbf{I}_n^{(\mathbf{s})}$ is the set

$$\text{Asc } \mathbf{e} = \left\{ i \in \{0, 1, \dots, n-1\} \mid \frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}} \right\}, \quad (1)$$

with the convention that $e_0 = 0$ (and $s_0 = 1$). The *ascent statistic* on $\mathbf{e} \in \mathbf{I}_n^{(\mathbf{s})}$ is

$$\text{asc } \mathbf{e} = |\text{Asc } \mathbf{e}|.$$

The *\mathbf{s} -Eulerian polynomial* is the generating polynomial for the ascent statistic on \mathbf{s} -inversion sequences $\mathbf{I}_n^{(\mathbf{s})}$:

$$\mathbf{E}_n^{(\mathbf{s})}(x) = \sum_{\mathbf{e} \in \mathbf{I}_n^{(\mathbf{s})}} x^{\text{asc } \mathbf{e}}.$$

It was shown in [18], Lemma 1, that

$$\mathbf{E}_n^{(1,2,\dots,n)}(x) = A_n(x) := \sum_{\pi \in \mathfrak{S}_n} x^{\text{des } \pi}.$$

Here $A_n(x)$ is the familiar *Eulerian polynomial* and $\text{des } \pi$ is the number of indices $i \in \{1, 2, \dots, n-1\}$ such that $\pi_i > \pi_{i+1}$. In addition to its many properties, $A_n(x)$ is known to have *only real roots* [13], a property which implies that its coefficient sequence is unimodal and log-concave.

Our main result is the following.

Theorem 1.1. *Let \mathbf{s} be any sequence of positive integers and n a positive integer. Then*

$$\mathbf{E}_n^{(\mathbf{s})}(x) = \sum_{\mathbf{e} \in \mathbf{I}_n^{(\mathbf{s})}} x^{\text{asc } \mathbf{e}}$$

has only real roots.

In Section 2, we prove Theorem 1.1 by the method of *compatible polynomials* using a result of Chudnovsky and Seymour [9].

Variations of the Eulerian polynomials arise as descent polynomials in families other than permutations. As we show in Section 3, Theorem 1.1 generalizes many previous results concerning the real-rootedness of these Eulerian polynomials and implies some new ones.

In Section 4, we discuss the geometric significance of Theorem 1.1. The *\mathbf{s} -lecture hall polytope*, $\mathcal{P}_n^{(\mathbf{s})}$ is defined by

$$\mathcal{P}_n^{(\mathbf{s})} = \left\{ \lambda \in \mathbb{R}^n \mid 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \dots \leq \frac{\lambda_n}{s_n} \leq 1 \right\}.$$

It was shown in [18, Theorem 5], that the *Ehrhart series* of $\mathcal{P}_n^{(\mathbf{s})}$ is

$$\frac{\mathbf{E}_n^{(\mathbf{s})}(x)}{(1-x)^{n+1}}.$$

In the language of Stanley [22], this says that $\mathbf{E}_n^{(\mathbf{s})}(x)$ is the *\mathbf{h}^* -polynomial* of $\mathcal{P}_n^{(\mathbf{s})}$.

By Theorem 1.1, the *\mathbf{h}^* -polynomial* of every *\mathbf{s} -lecture hall polytope* has all roots real. As a consequence, its coefficient sequence is unimodal and log-concave. This is not true of lattice polytopes in general, not even for 3-dimensional lattice simplices.

In Section 5, we extend Theorem 1.1 to a (q, z) -analog of $\mathbf{E}_n^{(\mathbf{s})}(x)$. With this, we show, for the first time, that the MacMahon–Carlitz *q -Eulerian polynomial* has only real roots for positive real

q , a result conjectured by Chow and Gessel in [7]. We further show that several other q -Eulerian polynomials for signed permutations and the wreath products $C_k \wr \mathfrak{S}_n$ (colored permutations, indexed permutations) are real-rooted for positive q . This includes the generating polynomial for the joint distribution of descent and flag-inversion number.

In Section 6, we focus of the generating polynomial for the joint distribution of descent and flag-major index on signed permutations and the wreath products $C_k \wr \mathfrak{S}_n$. We prove that this q -analog also has all roots real for positive q , a result which was conjectured by Chow and Gessel [7] for signed permutations and by Chow and Mansour [8], for $C_k \wr \mathfrak{S}_n$.

Additional applications and directions are discussed in Section 7.

2 The main result

In this section we prove Theorem 1.1 using the method of “compatible polynomials” in conjunction with a recurrence for $\mathbf{E}_n^{(s)}(x)$.

2.1 Compatible polynomials

Polynomials $f_1(x), \dots, f_m(x)$ over \mathbb{R} are *compatible* if, for all real $c_1, \dots, c_m \geq 0$, the polynomial $\sum_{i=1}^m c_i f_i(x)$ has only real roots.

We call such a sum $\sum_{i=1}^m c_i f_i(x)$ of polynomials, with $c_1, \dots, c_m \geq 0$ a *conic combination* of $f_1(x), \dots, f_m(x)$. A real-rooted polynomial (over \mathbb{R}) is compatible with itself.

The polynomials $f_1(x), \dots, f_m(x)$ are *pairwise compatible* if for all $i, j \in \{1, 2, \dots, m\}$, $f_i(x)$ and $f_j(x)$ are compatible.

Remark 2.1. *Polynomials $f(x)$ and $g(x)$ are compatible if and only if each of the following pairs is compatible*

- $af(x)$ and $bg(x)$ for any positive $a, b \in \mathbb{R}$
- $(c + dx)f(x)$ and $(c + dx)g(x)$ for any nonnegative $c, d \in \mathbb{R}$
- $f(qx)$ and $g(qx)$ for any positive $q \in \mathbb{R}$.

The following lemma is useful in proving that a collection of polynomials is compatible.

Lemma 2.2. [Chudnovsky-Seymour [9], 2.2] *The polynomials $f_1(x), \dots, f_m(x)$ with positive leading coefficients are pairwise compatible if and only if $f_1(x), \dots, f_m(x)$ are compatible.*

2.2 A recurrence for the s-Eulerian polynomial

In order to show that the s-Eulerian polynomial has all real roots, consider a refinement, where $0 \leq i < s_n$:

$$P_{n,i}^{(s)}(x) = \sum_{\{\mathbf{e} \in \mathbf{I}_n^{(s)} \mid e_n = i\}} x^{\text{asc } \mathbf{e}}. \quad (2)$$

Clearly,

$$\mathbf{E}_n^{(\mathbf{s})}(x) = \sum_{i=0}^{s_n-1} P_{n,i}^{(\mathbf{s})}(x). \quad (3)$$

We now prove a recurrence for $P_{n,i}^{(\mathbf{s})}(x)$.

Lemma 2.3. *Given sequence $\mathbf{s} = \{s_i\}_{i \geq 1}$ of positive integers, let $n \geq 1$ and $0 \leq i < s_n$. Then*

$$P_{n,i}^{(\mathbf{s})}(x) = \sum_{j=0}^{\ell-1} x P_{n-1,j}^{(\mathbf{s})}(x) + \sum_{j=\ell}^{s_{n-1}-1} P_{n-1,j}^{(\mathbf{s})}(x), \quad (4)$$

for $n > 1$, where

$$\ell = \lceil i s_{n-1} / s_n \rceil.$$

When $n = 1$, $P_{1,0}^{(\mathbf{s})}(x) = 1$ and $P_{1,i}^{(\mathbf{s})}(x) = x$ for $i > 0$.

Proof. For an inversion sequence $\mathbf{e} = (e_1, \dots, e_n) \in \mathbf{I}_n^{(\mathbf{s})}$ with $e_n = i$, let $j = e_{n-1}$. Then, by definition (1), $n-1 \in \text{Asc } \mathbf{e}$ if and only if $j/s_{n-1} < i/s_n$, or, equivalently, if $0 \leq j \leq \ell-1$ holds. So,

$$x^{\text{asc } \mathbf{e}} = \begin{cases} x^{1+\text{asc}(e_1, \dots, e_{n-1})} & \text{if } 0 \leq j \leq \ell-1, \\ x^{\text{asc}(e_1, \dots, e_{n-1})} & \text{if } \ell \leq j < s_{n-1}, \end{cases}$$

which proves (4). For the initial conditions, recall that $e_0/s_0 = 0$, by definition, and hence $0 \in \text{Asc } \mathbf{e}$ if and only if $e_1 > 0$. \square

2.3 Proof of Theorem 1.1

We will prove Theorem 1.1 by establishing the following—more general—theorem. Theorem 1.1 then follows in view of (3) and Lemma 2.2.

Theorem 2.4. *Given a sequence $\mathbf{s} = \{s_i\}_{i \geq 1}$ of positive integers, let $P_{n,i}^{(\mathbf{s})}(x)$ denote the polynomial defined in (2). Then for all $0 \leq i \leq j < s_n$,*

(i) *the polynomials $P_{n,i}^{(\mathbf{s})}(x)$ and $P_{n,j}^{(\mathbf{s})}(x)$ are compatible, and*

(ii) *the polynomials $x P_{n,i}^{(\mathbf{s})}(x)$ and $P_{n,j}^{(\mathbf{s})}(x)$ are compatible.*

As a consequence, $P_{n,i}^{(\mathbf{s})}(x)$ is real-rooted for $0 \leq i < s_n$.

Proof. We use induction on n . When $n = 1$, for $0 \leq i \leq j < s_1$, $(P_{1,i}^{(\mathbf{s})}(x), P_{1,j}^{(\mathbf{s})}(x)) \in \{(1,1), (1,x), (x,x)\}$ and thus $(x P_{1,i}^{(\mathbf{s})}(x), P_{1,j}^{(\mathbf{s})}(x)) \in \{(x,1), (x,x), (x^2,x)\}$. Clearly, each of the pairs of polynomials $(1,1), (1,x), (x,x), (x^2,x)$, is compatible.

Let $n > 1$, and assume that the theorem holds for all positive integers less than n . Let

$$\begin{aligned} \ell &= \lceil i s_n / s_{n-1} \rceil \\ k &= \lceil j s_n / s_{n-1} \rceil. \end{aligned}$$

Then $\ell \leq k$, since $i \leq j$ by assumption, and

$$P_{n,i}^{(s)}(x) = x \underbrace{(P_{n-1,0}^{(s)}(x) + \cdots + P_{n-1,\ell-1}^{(s)}(x))}_{\ell} + \cdots + P_{n-1,k-1}^{(s)}(x) + \cdots + P_{n-1,s_{n-1}}^{(s)}(x), \quad (5)$$

$$P_{n,j}^{(s)}(x) = x \underbrace{(P_{n-1,0}^{(s)}(x) + \cdots + P_{n-1,\ell-1}^{(s)}(x) + \cdots + P_{n-1,k-1}^{(s)}(x))}_{k} + \cdots + P_{n-1,s_{n-1}}^{(s)}(x). \quad (6)$$

We first show (i), i.e., that the polynomial $c_i P_{n,i}^{(s)} + c_j P_{n,j}^{(s)}$ has only real roots for all $c_i, c_j \geq 0$. According to (5) and (6), $c_i P_{n,i}^{(s)} + c_j P_{n,j}^{(s)}$ can be written as a conic combination of the following polynomials, which we group into three (possibly empty) sets:

$$\{x P_{n-1,\alpha}^{(s)} \mid 0 \leq \alpha < \ell\} \cup \{(c_i + c_j x) P_{n-1,\beta}^{(s)} \mid \ell \leq \beta < k\} \cup \{P_{n-1,\gamma}^{(s)} \mid k \leq \gamma \leq s_{n-1}\}.$$

Therefore, it suffices to show that these polynomials are compatible. By Lemma 2.2, it is equivalent to show that they are pairwise compatible.

Two polynomials from the same sets are compatible, by the induction hypothesis (i) and Remark 2.1. A polynomial from the first set is compatible with another from the third set by the induction hypothesis (ii).

To show compatibility between a polynomial from the first set and one from the second, we need that $ax P_{n-1,\alpha}^{(s)} + b(c_i + c_j x) P_{n-1,\beta}^{(s)}$ has only real roots for all $a, b, c_i, c_j \geq 0$ and $\alpha < \beta$. This expression is a conic combination of the $x P_{n-1,\alpha}^{(s)}$, $x P_{n-1,\beta}^{(s)}$, and $P_{n-1,\beta}^{(s)}$. Since $\alpha < \beta$, these three polynomials are pairwise compatible by induction, and hence compatible, by Lemma 2.2. The compatibility of a polynomial in the second set and one in the third set follows by a similar argument, exploiting the fact that, by induction, $x P_{n-1,\beta}^{(s)}$, $P_{n-1,\beta}^{(s)}$, and $P_{n-1,\gamma}^{(s)}$ are pairwise compatible for $\beta < \gamma$.

Now we are left to show (ii) that $x P_{n,i}^{(s)}$ and $P_{n,j}^{(s)}$ are compatible for $i < j$. In view of (5) and (6), in order to show that $c_i x P_{n,i}^{(s)} + c_j P_{n,j}^{(s)}$ is real-rooted for all $c_i, c_j \geq 0$ it suffices to show that

$$\begin{aligned} &\{x(c_i x + c_j) P_{n-1,\alpha}^{(s)} \mid 0 \leq \alpha < \ell\} \cup \{(c_i + c_j x) P_{n-1,\beta}^{(s)} \mid \ell \leq \beta < k\} \\ &\cup \{(c_i x + c_j) P_{n-1,\gamma}^{(s)} \mid k \leq \gamma \leq s_{(n-1)}\} \end{aligned}$$

is a set of compatible polynomials. This follows from analogous reasoning to the above. Two polynomials from the same subsets are compatible by Remark 2.1 and the induction hypothesis (i). Considering one from the first and one from the third: $x P_{n-1,\alpha}^{(s)}$ and $P_{n-1,\gamma}^{(s)}$ are compatible by induction. Multiplying both by the same linear factor $c_i x + c_j$, where $c_i, c_j \geq 0$, preserves compatibility by Remark 2.1. Similarly, $x^2 P_{n-1,\alpha}^{(s)}$, $x P_{n-1,\alpha}^{(s)}$, and $x P_{n-1,\beta}^{(s)}$ are pairwise compatible which settles the case when we have a polynomial from the first and one from the second subset. Finally, $x P_{n-1,\beta}^{(s)}$, $x P_{n-1,\gamma}^{(s)}$, and $P_{n-1,\gamma}^{(s)}$ are compatible, settling the case of one polynomial from the second subset and one from the third. \square

3 Applications

In this section, we show that Theorem 1.1 contains as special cases several existing real-rootedness results on (generalized) Eulerian polynomials, as well as some results which appear to be new.

3.1 Permutations and inversion sequences

We first show that Theorem 1.1 implies the real-rootedness of the familiar Eulerian polynomials, a result known since Frobenius [13].

Clearly, $|\mathbf{I}_n^{(1,2,\dots,n)}| = |\mathfrak{S}_n|$. For $\pi \in \mathfrak{S}_n$, let $\text{Des } \pi$ be the descent set of π ,

$$\text{Des } \pi = \{i \in \{1, \dots, n-1\} \mid \pi_i > \pi_{i+1}\},$$

and let $\text{inv } \pi$ be the number of *inversions* of π :

$$\text{inv } \pi = |\{(i, j) \mid 1 \leq i < j \leq n \text{ and } \pi_i > \pi_j\}|.$$

We will make use of the following bijection between \mathfrak{S}_n and $\mathbf{I}_n^{(1,2,\dots,n)}$ which was proved in [18, Lemma 1] to have the properties claimed.

Lemma 3.1. *The mapping $\phi : \mathfrak{S}_n \rightarrow \mathbf{I}_n^{(1,2,\dots,n)}$ defined by $\phi(\pi) = t = (t_1, t_2, \dots, t_n)$, where*

$$t_i = |\{j \in \{1, 2, \dots, i-1\} \mid \pi_j > \pi_i\}|$$

is a bijection satisfying both $\text{Des } \pi = \text{Asc } t$ and $\text{inv } \pi = |t| := t_1 + t_2 + \dots + t_n$.

Corollary 3.2. *For $n \geq 1$, the Eulerian polynomial,*

$$\sum_{\pi \in \mathfrak{S}_n} x^{\text{des } \pi} := A_n(x),$$

has only real roots.

Proof. By Lemma 3.1,

$$A_n(x) := \sum_{\pi \in \mathfrak{S}_n} x^{\text{des } \pi} = \sum_{\mathbf{e} \in \mathbf{I}_n^{(1,2,\dots,n)}} x^{\text{asc } \mathbf{e}} := \mathbf{E}_n^{(1,2,\dots,n)}(x),$$

which has all roots real by Theorem 1.1 with $\mathbf{s} = (1, 2, \dots, n)$. □

3.2 Signed permutations

Let \mathfrak{B}_n denote the hyperoctahedral group, whose elements are the signed permutations of $\{1, 2, \dots, n\}$. Each $\sigma \in \mathfrak{B}_n$ has the form $(\pm\pi_1, \pm\pi_2, \dots, \pm\pi_n)$ where $\pi \in \mathfrak{S}_n$.

In defining the notion of “descent” on \mathfrak{B}_n , both of the orderings

$$-n < -(n-1) < \dots < -1 < 0 < 1 < 2 < \dots < n$$

and

$$-1 < -2 < \dots < -n < 0 < 1 < 2 < \dots < n$$

have been used in the literature. We will assume the second ordering, since it generalizes naturally to the wreath products in the next subsection.

An index $i \in \{0, 1, \dots, n-1\}$ is a *descent* of σ if $\sigma_i > \sigma_{i+1}$. By convention, $\sigma_0 = 0$. Let $\text{des}_{\mathfrak{B}} \sigma$ denote the number of descents of σ .

There is a correspondence between statistics on \mathfrak{B}_n and statistics on inversion sequences. The following was shown in [18, eq. (26)].

Lemma 3.3.

$$\sum_{t \geq 0} (2t+1)^n x^t = \frac{\mathbf{E}_n^{(2,4,\dots,2n)}(x)}{(1-x)^{n+1}} = \frac{\sum_{e \in \mathbf{I}_n^{(2,4,\dots,2n)}} e^{\text{asc } e}}{(1-x)^{n+1}}. \quad (7)$$

On the other hand, the infinite series in (7) was shown by Steingrímsson [24] to satisfy:

$$\sum_{t \geq 0} (2t+1)^n x^t = \frac{\sum_{\pi \in \mathfrak{B}_n} x^{\text{des}_B(\pi)}}{(1-x)^{n+1}}. \quad (8)$$

So, we have the following result, originally due to Brenti [4, Corollary 3.7].

Corollary 3.4. *The descent polynomial for signed permutations,*

$$B_n(x) := \sum_{\pi \in \mathfrak{B}_n} x^{\text{des}_B(\pi)},$$

has all real roots.

Proof. Combining Lemma 3.3 and (8),

$$B_n(x) := \sum_{\pi \in \mathfrak{B}_n} x^{\text{des}_B(\pi)} = \sum_{e \in \mathbf{I}_n^{(2,4,\dots,2n)}} e^{\text{asc } e} := \mathbf{E}_n^{(2,4,\dots,2n)}(x).$$

The result follows with $\mathbf{s} = (2, 4, \dots, 2n)$ in Theorem 1.1. \square

3.3 The wreath products $C_k \wr \mathfrak{S}_n$

For positive integer k , the wreath product $C_k \wr \mathfrak{S}_n$, of a cyclic group, C_k , of order k , and the symmetric group \mathfrak{S}_n , generalizes both \mathfrak{S}_n (the case $k = 1$) and \mathfrak{B}_n ($k = 2$).

We regard $C_k \wr \mathfrak{S}_n$ as the set of pairs (π, σ) , written as

$$\pi^\sigma = (\pi_1^{\sigma_1}, \pi_2^{\sigma_2}, \dots, \pi_n^{\sigma_n}),$$

where $\pi = (\pi_1, \dots, \pi_n) \in \mathfrak{S}_n$ and $\sigma = (\sigma_1, \dots, \sigma_n) \in \{0, 1, \dots, k-1\}^n$.

The *descent set* of $\pi^\sigma \in C_k \wr \mathfrak{S}_n$ is

$$\text{Des } \pi^\sigma = \{i \in \{0, 1, \dots, n-1\} \mid \sigma_i < \sigma_{i+1} \text{ or } \sigma_i = \sigma_{i+1} \text{ and } \pi_i > \pi_{i+1}\}, \quad (9)$$

with the convention that $\pi_0 = \sigma_0 = 0$. Let $\text{des } \pi^\sigma = |\text{Des } \pi^\sigma|$. Note that this definition of des agrees with des on \mathfrak{S}_n when $k = 1$, and with des_B on \mathfrak{B}_n when $k = 2$.

The descent polynomial for $C_k \wr \mathfrak{S}_n$ is

$$W_{n,k}(x) := \sum_{\pi^\sigma \in C_k \wr \mathfrak{S}_n} x^{\text{des } \pi^\sigma}.$$

It is known that $W_{n,k}(x)$ has all real roots, a result originally due to Steingrímsson [23, Theorem 3.19]. We show that it also follows from Theorem 1.1.

It was shown in [17] that statistics on $C_k \wr \mathfrak{S}_n$ are related to statistics on \mathbf{s} -inversion sequences, $\mathbf{I}_n^{(\mathbf{s})}$, where $\mathbf{s} = (k, 2k, \dots, nk)$, as we now describe.

Let $\mathbf{I}_{n,k} = \mathbf{I}_n^{(k, 2k, \dots, nk)}$. As sets, $\mathbf{I}_{n,k}$ and $C_k \wr \mathfrak{S}_n$ have the same cardinality. The following bijection was proven in [17, Theorem 3] to map “des” on $C_k \wr \mathfrak{S}_n$ to “asc” on $\mathbf{I}_{n,k}$.

Lemma 3.5. *For each pair (n, k) with $n \geq 1$, $k \geq 1$, define*

$$\Theta : C_k \wr \mathfrak{S}_n \longrightarrow \mathbf{I}_{n,k}$$

by

$$e = \Theta(\pi_1^{\sigma_1}, \pi_2^{\sigma_2}, \dots, \pi_n^{\sigma_n}) = (\sigma_1 + t_1, 2\sigma_2 + t_2, \dots, n\sigma_n + t_n), \quad (10)$$

where $(t_1, t_2, \dots, t_n) = \phi(\pi)$, for ϕ defined on \mathfrak{S}_n as in Lemma 3.1.

Then

$$\text{Asc } e = \text{Des } \pi^\sigma.$$

Corollary 3.6. *For each pair (n, k) with $n \geq 1$, $k \geq 1$, the descent polynomial of $C_k \wr \mathfrak{S}_n$ has all roots real.*

Proof. By Lemma 3.5,

$$W_{n,k}(x) := \sum_{\pi^\sigma \in C_k \wr \mathfrak{S}_n} x^{\text{des } \pi^\sigma} = \sum_{\mathbf{e} \in \mathbf{I}_{n,k}} x^{\text{asc } \mathbf{e}} := \mathbf{E}_n^{(k, 2k, \dots, nk)}(x),$$

so the result follows from Theorem 1.1 with $\mathbf{s} = (k, 2k, \dots, nk)$. \square

In Section 5 we will use the fact that the bijection Θ of Lemma 3.5 relates other statistics of $C_k \wr \mathfrak{S}_n$ and $\mathbf{I}_{n,k}$, to show that several q -analogs of $W_{n,k}(x)$ are real-rooted for all positive q , settling some open questions.

3.4 k -ary words

The k -ary words of length n are the elements of the set $\{0, 1, \dots, k-1\}^n$. Define an ascent statistic for $w \in \{0, 1, \dots, k-1\}^n$ by

$$\text{asc } w = \{i \in \{0, 1, \dots, n-1\} \mid w_i < w_{i+1}\},$$

with the convention that $w_0 = 0$.

Corollary 3.7. *The ascent polynomial for k -ary words,*

$$\sum_{w \in \{0, 1, \dots, k-1\}^n} x^{\text{asc } w},$$

has all real roots.

Proof. Clearly, using the identity mapping from $\{0, 1, \dots, k-1\}^n$ to $\mathbf{I}_n^{(k,k,\dots,k)}$,

$$\sum_{w \in \{0,1,\dots,k-1\}^n} x^{\text{asc } w} = \sum_{\mathbf{e} \in \mathbf{I}_n^{(k,k,\dots,k)}} x^{\text{asc } \mathbf{e}}.$$

So, the result follows by setting $\mathbf{s} = (k, k, \dots, k)$ in Theorem 1.1. \square

It was shown in [18, Corollary 8] using Ehrhart theory that

$$\frac{\sum_{\mathbf{e} \in \mathbf{I}_n^{(k,2k,\dots,nk)}} x^{\text{asc } \mathbf{e}}}{(1-x)^{n+1}} = \sum_{t \geq 0} \binom{n+kt}{n} x^t.$$

3.5 Excedances and number of cycles in permutations

For a permutation $\pi \in \mathfrak{S}_n$, the *excedance* number of π , $\text{exc}(\pi)$, is defined by

$$\text{exc}(\pi) = |\{i \in \{1, 2, \dots, n\} \mid \pi(i) > i\}|,$$

and the *cycle number* of π , $\text{cyc}(\pi)$, is the number of cycles in the disjoint cycle representation of π . Let

$$F_n(x, y) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{exc } \pi} y^{\text{cyc } \pi}.$$

It was proven by Brenti in [5, Theorem 7.5] that $F_n(x, y)$ has all roots real for every positive $y \in \mathbb{R}$. This was extended by Brändén in [3, Theorem 6.3] to include values of y for which $n + y \leq 0$.

In [19], $F_n(x, 1/k)$ was shown to be related to inversion sequences. This will allow us to deduce the real-rootedness in this special case from Theorem 1.1.

Corollary 3.8. *For every positive integer, k , the polynomial*

$$F_n(x, 1/k) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{exc } \pi} k^{-\text{cyc } \pi}$$

has only real roots.

Proof. Let $\mathbf{s} = (k+1, 2k+1, \dots, (n-1)k+1)$. It was shown in [19, Theorems 3 and 6], that for every positive integer k ,

$$\sum_{\mathbf{e} \in \mathbf{I}_n^{(\mathbf{s})}} x^{\text{asc } \mathbf{e}} = k^n F_n(x, 1/k).$$

The corollary follows from Theorem 1.1 with $\mathbf{s} = (k+1, 2k+1, \dots, (n-1)k+1)$. \square

3.6 Multiset permutations

Rodica Simion [20, Section 2] showed that for any n -element multiset, M , the descent polynomial for the set of permutations, $P(M)$, of M has only real roots. A *descent* in a multiset permutation, $\pi \in P(M)$ is an index $i \in \{1, 2, \dots, n-1\}$ such that $\pi_i > \pi_{i+1}$.

When $M = \{1, 1, 2, 2, \dots, n, n\}$, there is a connection with inversion sequences. Let \mathbf{s} be the sequence $\mathbf{s} = (1, 1, 3, 2, 5, 3, 7, 4, \dots)$, where for $i \geq 1$, $s_{2i} = i$ and $s_{2i-1} = 2i - 1$. Observe that the number of \mathbf{s} -inversion sequences of length $2n$ is the same as the number of permutations of $\{1, 1, 2, 2, \dots, n, n\}$:

$$\left| I_{2n}^{(1,1,3,2,5,3,7,4,\dots,2n-1,n)} \right| = \frac{(2n)!}{2^n} = |P(\{1, 1, 2, 2, \dots, n, n\})|.$$

We discovered that the distribution of ascents on the first set is equal to the distribution of descents on the second set.

Theorem 3.9.

$$\sum_{\pi \in P(\{1,1,2,2,\dots,n,n\})} x^{\text{des } \pi} = \sum_{\mathbf{e} \in \mathbf{I}_{2n}^{(1,1,3,2,5,3,\dots,2n-1,n)}} x^{\text{asc } \mathbf{e}}.$$

Proof. It was shown in [18, Theorem 14] that

$$\sum_{t \geq 0} \left(\frac{(t+1)(t+2)}{2} \right)^n x^t = \frac{\sum_{\mathbf{e} \in \mathbf{I}_{2n}^{(1,1,3,2,5,3,\dots,2n-1,n)}} x^{\text{asc } \mathbf{e}}}{(1-x)^{2n+1}}.$$

MacMahon [15, Volume 2, Chapter IV, p. 211, §462] showed that

$$\frac{\sum_{\pi \in P(\{1^{p_1}, 2^{p_2}, \dots, n^{p_n}\})} x^{\text{des } \pi}}{(1-x)^{1+\sum p_i}} = \sum_{t \geq 0} \frac{(t+1) \cdots (t+p_1) \cdots (t+1) \cdots (t+p_n)}{p_1! \cdot p_2! \cdots p_n!} x^t.$$

In particular, when $p_i = 2$ for all i , this implies

$$\frac{\sum_{\pi \in P(\{1,1,2,2,\dots,n,n\})} x^{\text{des } \pi}}{(1-x)^{2n+1}} = \sum_{t \geq 0} \left(\frac{(t+1)(t+2)}{2} \right)^n x^t.$$

□

We thus obtain the following special case of Simion's result as a corollary of Theorem 1.1.

Corollary 3.10. *The polynomial*

$$\sum_{\pi \in P(\{1,1,2,2,\dots,n,n\})} x^{\text{des } \pi}$$

has only real roots.

The sequence $\mathbf{s} = (1, 1, 3, 2, 5, 3, 7, 4, \dots)$ was studied in [10], where it was shown that the \mathbf{s} -lecture hall partitions lead to a new finite model for the *Little Göllnitz identities*. There was a companion sequence, $\mathbf{s} = (1, 4, 3, 8, 5, 12, \dots, 2n-1, 4n)$ defined by $s_{2i} = 4i$, $s_{2i+1} = 2i+1$, which we now consider in the context of multiset permutations.

Let $P^\pm(\{1, 1, 2, 2, \dots, n, n\})$ be the set of all signed permutations of $\{1, 1, 2, 2, \dots, n, n\}$. The elements are those of the form $(\pm\pi_1, \pm\pi_2, \dots, \pm\pi_{2n})$, where $(\pi_1, \pi_2, \dots, \pi_{2n}) \in P(\{1, 1, 2, 2, \dots, n, n\})$. Note that

$$|P^\pm(\{1, 1, 2, 2, \dots, n, n\})| = \frac{(2n)!}{2^n} 2^{2n} = 2^n (2n)! = |I_{2n}^{(1,4,3,8,5,12,\dots,2n-1,4n)}|.$$

From our experiments it appears that distribution of descents on the first set is equal to the distribution of ascents on the second set, and we make that conjecture.

Conjecture 3.11.

$$\sum_{\pi \in P^\pm(\{1,1,2,2,\dots,n,n\})} x^{\text{des } \pi} = \sum_{\mathbf{e} \in \mathbf{I}_{2n}^{(1,4,3,8,5,12,\dots,2n-1,4n)}} x^{\text{asc } \mathbf{e}}.$$

If this conjecture is true, it would follow as a corollary of Theorem 1.1 that the descent polynomial for the signed permutations of the multiset $\{1, 1, 2, 2, \dots, n, n\}$ has all real roots.

It was shown in [18, Theorem 13] that

$$\sum_{t \geq 0} ((t+1)(2t+1))^n x^t = \frac{\sum_{\mathbf{e} \in \mathbf{I}_{2n}^{(1,4,3,8,5,12,\dots,2n-1,4n)}} x^{\text{asc } \mathbf{e}}}{(1-x)^{2n+1}}.$$

It may be possible to show that $\sum_{\pi \in P^\pm(\{1,1,2,2,\dots,n,n\})} x^{\text{des } \pi}$ satisfies the same identity.

4 The h^* -polynomials of s-lecture hall polytopes

In this section we describe a geometric consequence of Theorem 1.1.

For background, the *Ehrhart series* of a polytope \mathcal{P} in \mathbb{R}^n is the series

$$\sum_{t \geq 0} |t\mathcal{P} \cap \mathbb{Z}^n| x^t,$$

where $t\mathcal{P}$ is the t -fold *dilation* of \mathcal{P} :

$$t\mathcal{P} = \{(t\lambda_1, t\lambda_2, \dots, t\lambda_n) \mid (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathcal{P}\}.$$

So, $i(\mathcal{P}, t) := |t\mathcal{P} \cap \mathbb{Z}^n|$ is the number of points in $t\mathcal{P}$, all of whose coordinates are integer. For example, if

$$\mathcal{P} = \{\lambda \in \mathbb{R}^3 \mid \lambda_1 \leq 2\lambda_2, \lambda_2 \leq 2\lambda_1, \text{ and } \lambda_1 + \lambda_2 \leq 2\}, \quad (11)$$

then

$$i(\mathcal{P}, t) = |t\mathcal{P} \cap \mathbb{Z}^n| = 1 + 3/2t + 3/2t^2,$$

and the Ehrhart series of \mathcal{P} is

$$\sum_{t \geq 0} (1 + 3/2t + 3/2t^2) x^t = \frac{x^2 + x + 1}{(1-x)^3}. \quad (12)$$

If all vertices of a polytope \mathcal{P} are integer, then $i(\mathcal{P}, t)$ is a *polynomial* in t and the Ehrhart series of \mathcal{P} has the form

$$\sum_{t \geq 0} i(\mathcal{P}, t) x^t = \frac{\mathbf{h}(x)}{(1-x)^n},$$

for a polynomial $\mathbf{h}(x) = h_0 + h_1 x + \cdots + h_d x^d$, known as the \mathbf{h}^* -polynomial of \mathcal{P} [11, 12]. Here d is the dimension of \mathcal{P} .

By Stanley's *Nonnegativity Theorem* [22], if \mathcal{P} is a convex polytope with integer vertices, the coefficients of its \mathbf{h}^* -polynomial are nonnegative. The sequence of coefficients $[h_1, h_2, \dots, h_d]$ of $\mathbf{h}(x)$ is called the \mathbf{h}^* -vector of \mathcal{P} .

For example, in the case of the polytope \mathcal{P} , defined by (11), the \mathbf{h}^* -polynomial is $\mathbf{h}(x) = x^2 + x + 1$ and the \mathbf{h}^* -vector is $[1, 1, 1]$, which is nonnegative, symmetric, and unimodal.

The \mathbf{h}^* -vector of a convex polytope with integer vertices need not be symmetric or unimodal. Although there has been much progress in the direction of characterizing those polytopes whose \mathbf{h}^* -vector is unimodal (see, e.g., [2, 6, 16, 21]) this is still an open question.

However, we can use Theorem 1.1 to answer the question for the following class of polytopes associated with \mathbf{s} -inversion sequences.

The \mathbf{s} -lecture hall polytope $\mathcal{P}_n^{(\mathbf{s})}$ is defined by

$$\mathcal{P}_n^{(\mathbf{s})} = \left\{ \lambda \in \mathbb{R}^n \mid 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \cdots \leq \frac{\lambda_n}{s_n} \leq 1 \right\},$$

where \mathbf{s} is an arbitrary sequence of positive integers.

The following is a special case of Theorem 5 in [18].

Lemma 4.1. *For any sequence \mathbf{s} of positive integers,*

$$\sum_{t \geq 0} i(\mathcal{P}_n^{(\mathbf{s})}, t) x^t = \frac{\mathbf{E}_n^{(\mathbf{s})}(x)}{(1-x)^{n+1}}.$$

So combining Lemma 4.1 with Theorem 1.1 we have:

Corollary 4.2. *For any sequence \mathbf{s} of positive integers, the \mathbf{h}^* -polynomial of the \mathbf{s} -lecture hall polytope has all roots real.*

The \mathbf{s} -lecture hall polytopes are special in this regard, even among lattice simplices. The polytope \mathcal{P} of the example (11) is a simplex in \mathbb{R}^3 with integer vertices, but its \mathbf{h}^* -polynomial, $x^2 + x + 1$, does not have real roots.

The sequence of coefficients of a real polynomial with only real roots is log-concave, and—if the coefficients are non-negative—it is also unimodal. This is an easy corollary of a classic result, often referred to as Newton's inequality. Thus, we have the following.

Corollary 4.3. *For any sequence \mathbf{s} of positive integers, The \mathbf{h}^* -vector of the \mathbf{s} -lecture hall polytope is unimodal and log-concave.*

Remark 4.4. *The \mathbf{h}^* -vector an \mathbf{s} -lecture hall polytope need not be symmetric. For example,*

$$\mathbf{E}_n^{(1,3,5)}(x) = 1 + 10x + 4x^2.$$

5 (q, z) -analogs of \mathbf{s} -Eulerian polynomials

In this section we define (q, z) -analogs of the \mathbf{s} -Eulerian polynomials show that they have real roots for every positive $q, z \in \mathbb{R}$.

As one special case, we show for the first time that the MacMahon–Carlitz q -Eulerian polynomial is real rooted for $q > 0$, a result conjectured by Chow and Gessel in [7].

In addition to the statistics $\text{Asc } \mathbf{e}$, $\text{asc } \mathbf{e}$ and $|\mathbf{e}| = \sum_i e_i$ on \mathbf{s} -inversion sequences, we define a new statistic, related to the major index on permutations. For $\mathbf{e} \in \mathbf{I}_n^{(\mathbf{s})}$, define

$$\text{amaj } \mathbf{e} = \sum_{j \in \text{Asc } \mathbf{e}} (n - j).$$

We will prove the following.

Theorem 5.1. *For \mathbf{s} a sequence of positive integers, let*

$$\mathbf{E}_n^{(\mathbf{s})}(x, q, z) = \sum_{\mathbf{e} \in \mathbf{I}_n^{(\mathbf{s})}} x^{\text{asc } \mathbf{e}} q^{\text{amaj } \mathbf{e}} z^{|\mathbf{e}|}.$$

For any positive real numbers q and z , $\mathbf{E}_n^{(\mathbf{s})}(x, q, z)$ has all real roots.

The proof uses the method of Section 2. For $n \geq 1$ and $0 \leq i < s_n$, define

$$P_{n,i}^{(\mathbf{s})}(x, q, z) = \sum_{\{\mathbf{e} \in \mathbf{I}_n^{(\mathbf{s})} \mid e_n = i\}} x^{\text{asc } \mathbf{e}} q^{\text{amaj } \mathbf{e}} z^{|\mathbf{e}|}.$$

Lemma 5.2. *For $n \geq 1$ and $0 \leq i < s_n$,*

$$P_{n,i}^{(\mathbf{s})}(x, q, z) = z^i \left(\sum_{j=0}^{\ell-1} xq P_{n-1,j}^{(\mathbf{s})}(xq, q, z) + \sum_{j=\ell}^{s_{n-1}-1} P_{n-1,j}^{(\mathbf{s})}(xq, q, z) \right)$$

where $\ell = \lceil is_{n-1}/s_n \rceil$, and with initial conditions $P_{1,0}^{(\mathbf{s})}(x, q, z) = 1$ and $P_{1,i}^{(\mathbf{s})}(x, q, z) = xqz^i$ for $i > 0$.

Proof. For $\mathbf{e} = (e_1, \dots, e_n) \in \mathbf{I}_n^{(\mathbf{s})}$ with $e_n = i$, let $j = e_{n-1}$. Then $n-1 \in \text{Asc } \mathbf{e}$ if and only if $j/s_{n-1} < i/s_n$, that is $0 \leq j \leq \ell-1$. Let $T_j = xq$ if $j \leq \ell-1$ and otherwise $T_j = 1$. Then

$$\begin{aligned} x^{\text{asc } \mathbf{e}} q^{\text{amaj } \mathbf{e}} z^{|\mathbf{e}|} &= z^{e_1+e_2+\dots+e_n} \prod_{t \in \text{Asc } \mathbf{e}} (xq^{n-t}) \\ &= z^i T_j z^{e_1+e_2+\dots+e_{n-1}} \prod_{t \in \text{Asc}(e_1, \dots, e_{n-1})} ((xq)q^{n-1-t}) \\ &= z^i T_j z^{e_1+e_2+\dots+e_{n-1}} (xq)^{\text{asc}(e_1, \dots, e_{n-1})} q^{\text{amaj}(e_1, \dots, e_{n-1})}. \end{aligned}$$

For the initial conditions, $0 \in \text{Asc } \mathbf{e}$ if and only if $e_1 > 0$, in which case $\text{des } \mathbf{e} = 1$, $\text{amaj } \mathbf{e} = n-0 = 1$ and $|\mathbf{e}| = e_1 = i$. \square

Since

$$\mathbf{E}_n^{(\mathbf{s})}(x, q, z) = \sum_{i=0}^{s_n-1} P_{n,i}(x, q, z),$$

Theorem 5.1 will follow if we prove that the polynomials $\{P_{n,i}(x, q, z) \mid 0 \leq i < s_n\}$ are compatible. We now prove this is true for any polynomials which satisfy a certain generalization of the recurrence of Lemma 5.2.

Theorem 5.3. *Let \mathbf{s} be any sequence of positive integers and k a nonnegative integer. Suppose, for $n > 1$ and $0 \leq i < s_n$, and $\ell = \lceil is_n/s_{n-1} \rceil$, that $R_{n,i}(x, q, z)$ is defined by*

$$R_{n,i}(x, q, z) = m_i \left(\sum_{j=0}^{\ell-1} xq^k R_{n-1,i}(xq^k, q, z) + \sum_{j=\ell}^{s_{n-1}-1} R_{n-1,i}(xq^k, q, z) \right),$$

with initial conditions $R_{1,0}(x, q, z) = b_0$, $R_{1,i}(x, q, z) = b_i x$, where $m_i = m_i(k, n, q, z)$ and $b_i = b_i(k, n, q, z)$ are independent of x and satisfy $m_i > 0$, $b_i > 0$ whenever $i, n, q, z > 0$ and $k \geq 0$.

Then for all $0 \leq i \leq j < s_n$, and for positive $q, z \in \mathbb{R}$,

- (i) the polynomials $R_{n,i}(x, q, z)$ and $R_{n,j}(x, q, z)$ are compatible, and
- (ii) the polynomials $xR_{n,i}(x, q, z)$ and $R_{n,j}(x, q, z)$ are compatible.

As a consequence, $R_{n,i}(x, q, z)$ is real-rooted for $q, z > 0$ and $0 \leq i < s_n$.

Proof. Fix $q, z > 0$. We use induction on n . When $n = 1$, for $0 \leq i \leq j < s_1$,

$$(R_{1,i}(x, q, z), R_{1,j}(x, q, z)) \in \{(b_0, b_0), (b_0, b_j x), (b_i x, b_j x)\}$$

and thus

$$(xR_{1,i}(x, q, z), R_{1,j}(x, q, z)) \in \{(b_0 x, b_0), (b_0 x, b_j x), (b_i x^2, b_j x)\}.$$

Clearly, each of the pairs of polynomials (b_0, b_0) , $(b_0, b_j x)$, $(b_i x, b_j x)$, $(b_0 x, b_0)$, $(b_i x^2, b_j x)$ is compatible.

When $n > 1$, we will apply the induction hypothesis. Observe that for $i \leq j$, since, by induction, $R_{n-1,i}(x, q, z)$ and $R_{n-1,j}(x, q, z)$ are compatible so also are $R_{n-1,i}(xq^k, q, z)$ and $R_{n-1,j}(xq^k, q, z)$ by Remark 2.1, since $q^k > 0$. Similarly, since $xR_{n-1,i}(x, q, z)$ and $R_{n-1,j}(x, q, z)$ are compatible by induction, so are $xq^k R_{n-1,i}(xq^k, q, z)$ and $R_{n-1,j}(xq^k, q, z)$ by Remark 2.1. By Remark 2.1, multiplying compatible polynomials by (possibly different) positive reals preserves compatibility. Therefore, the polynomials $xR_{n-1,i}(xq^k, q, z)$ and $R_{n-1,j}(xq^k, q, z)$ are also compatible. We will use this last observation in the proof of (ii).

To prove (i) we show that $c_i R_{n,i}(x, q, z) + c_j R_{n,j}(x, q, z)$ has only real roots for $i \leq j$ and $c_i, c_j \geq 0$. Let $k = \lceil js_n/s_{n-1} \rceil$, so $\ell \leq k$. Observe that, by the recursion of Lemma 5.2, $c_i R_{n,i}(x, q, z) + c_j R_{n,j}(x, q, z)$ can be written as a conic combination of polynomials from the following three (possibly empty) sets.

$$\{xq^k(c_i m_i + c_j m_j)R_{n-1,\alpha}(xq^k, q, z) \mid 0 \leq \alpha < \ell\},$$

$$\begin{aligned} & \{(c_i m_i + c_j m_j x q^k) R_{n-1, \beta}(x q^k, q, z) \mid \ell \leq \beta < k\}, \\ & \{(c_i m_i + c_j m_j) R_{n-1, \gamma}(x q^k, q, z) \mid k \leq \gamma < s_{n-1}\}. \end{aligned}$$

Two polynomials from the same set are compatible, by induction, as are one from the first set and one from the third. A conic combination of polynomials from the first and second set would be a conic combination of $x q^k R_{n-1, \alpha}(x q^k, q, z)$, $R_{n-1, \beta}(x q^k, q, z)$, and $x q^k R_{n-1, \beta}(x q^k, q, z)$, which are pairwise compatible, by induction, since $\alpha \leq \beta$. Similarly, A conic combination of polynomials from the second and third set would be a conic combination of $R_{n-1, \beta}(x q^k, q, z)$, $x q^k R_{n-1, \beta}(x q^k, q, z)$, and $R_{n-1, \gamma}(x q^k, q, z)$, which are pairwise compatible, by induction, since $\beta \leq \gamma$.

To prove (ii), by the recursion of Lemma 5.2, $c_i x R_{n, i}(x, q, z) + c_j R_{n, j}(x, q, z)$ can be written as a conic combination of polynomials from the following three (possibly empty) sets:

$$\begin{aligned} & \{x q^k (c_i m_i x + c_j m_j) R_{n-1, \alpha}(x q^k, q, z) \mid 0 \leq \alpha < \ell\}, \\ & \{x (c_i m_i + c_j m_j q^k) R_{n-1, \beta}(x q^k, q, z) \mid \ell \leq \beta < k\}, \\ & \{(c_i m_i x + c_j m_j) R_{n-1, \gamma}(x q^k, q, z) \mid k \leq \gamma < s_{n-1}\}. \end{aligned}$$

Two polynomials from the same set are compatible, by induction. Considering one from the first and one from the third: $x q^k R_{n-1, \alpha}(x q^k, q, z)$ and $R_{n-1, \gamma}(x q^k, q, z)$ are compatible by induction. Multiplying both by the same linear factor $c_i m_i x + c_j m_j$ preserves compatibility by Remark 2.1.

A conic combination of polynomials from the first and second set would be a conic combination of $x^2 q^k R_{n-1, \alpha}(x q^k, q, z)$, $x q^k R_{n-1, \alpha}(x q^k, q, z)$, $x R_{n-1, \beta}(x q^k, q, z)$, and $x q^k R_{n-1, \beta}(x q^k, q, z)$, which, by Remark 2.1 are compatible if and only if the polynomials $x^2 R_{n-1, \alpha}(x q^k, q, z)$, $x R_{n-1, \alpha}(x q^k, q, z)$, and $x R_{n-1, \beta}(x q^k, q, z)$ are compatible. As discussed at the beginning of this proof, by induction and by Remark 2.1, these polynomials are pairwise compatible.

Similarly, a conic combination of polynomials from the second and third sets would be a conic combination of $x R_{n-1, \beta}(x q^k, q, z)$, $x q^k R_{n-1, \beta}(x q^k, q, z)$, $x R_{n-1, \gamma}(x q^k, q, z)$, and $R_{n-1, \gamma}(x q^k, q, z)$, which, by Remark 2.1 are compatible if and only if the polynomials $x R_{n-1, \beta}(x q^k, q, z)$, $x R_{n-1, \gamma}(x q^k, q, z)$, and $R_{n-1, \gamma}(x q^k, q, z)$ are compatible. By induction and by Remark 2.1, these polynomials are pairwise compatible, completing the proof. \square

Proof. of Theorem 5.1

For fixed \mathbf{s} , setting $k = 1$, $b_0 = 1$, $b_i = q z^i$, and $m_i = z^i$ in Theorem 5.3 gives the recurrence of Lemma 5.2. Thus, by Theorem 5.3, the polynomials $\{P_{n, i}(x, q, z) \mid 0 \leq i < s_n\}$ are pairwise compatible and so, by Lemma 2.2, they are compatible. This implies that the polynomial

$$\mathbf{E}_n^{(\mathbf{s})}(x, q, z) = \sum_{i=0}^{s_n-1} P_{n, i}^{(\mathbf{s})}(x, q, z)$$

has all real roots. \square

We apply Theorem 5.1 now to the case of permutations. For $\pi \in \mathfrak{S}_n$, let

$$\begin{aligned} \text{maj } \pi &= \sum_{j \in \text{Des } \pi} j; \\ \text{comaj } \pi &= \sum_{j \in \text{Des } \pi} (n - j). \end{aligned}$$

It was conjectured in [7], that for any $q > 0$, the polynomial $A_n(x, q)$, defined by

$$A_n(x, q) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{des } \pi} q^{\text{maj } \pi},$$

has all real roots for $q > 1$. We now settle this conjecture as part of the following theorem.

Theorem 5.4. *For the usual Eulerian polynomials $A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{des } \pi}$, all of the following q -analogs are real-rooted when $q > 0$:*

$$A_n(x, q) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{des } \pi} q^{\text{inv } \pi}, \quad (13)$$

$$A_n(x, q) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{des } \pi} q^{\text{comaj } \pi}, \quad (14)$$

$$A_n(x, q) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{des } \pi} q^{\text{maj } \pi}. \quad (15)$$

The last of these is known as the MacMahon-Carlitz q -Eulerian polynomial.

Proof. The bijection $\phi : \mathfrak{S}_n \rightarrow \mathbf{I}_n^{(1,2,\dots,n)}$ of Lemma 3.1 maps (Des, inv) on \mathfrak{S}_n to $(\text{Asc}, | \cdot |)$ on $\mathbf{I}_n^{(1,2,\dots,n)}$, so we have

$$\mathbf{E}_n^{(1,2,\dots,n)}(x, q, z) = \sum_{\mathbf{e} \in \mathbf{I}_n^{(1,2,\dots,n)}} x^{\text{asc } \mathbf{e}} q^{\text{amaj } \mathbf{e}} z^{|\mathbf{e}|} = \sum_{\pi \in \mathfrak{S}_n} x^{\text{des } \pi} q^{\text{comaj } \pi} z^{\text{inv } \pi}.$$

So, $\mathbf{E}_n^{(1,2,\dots,n)}(x, 1, q)$ and $\mathbf{E}_n^{(1,2,\dots,n)}(x, q, 1)$ are the polynomials (13) and (14), respectively, and by Theorem 5.1 they are real-rooted for $q > 1$. As for (15), observe that the mapping $\mathfrak{S}_n \rightarrow \mathfrak{S}_n$ defined by

$$\pi = (\pi_1, \pi_2, \dots, \pi_n) \rightarrow \pi' = (n+1-\pi_n, n+1-\pi_{n-1}, \dots, n+1-\pi_1)$$

satisfies $\text{Des } \pi' = \{n-j \mid j \in \text{Des } \pi\}$. Thus

$$\sum_{\pi \in \mathfrak{S}_n} x^{\text{des } \pi} q^{\text{maj } \pi} = \sum_{\pi' \in \mathfrak{S}_n} x^{\text{des } \pi'} q^{\text{comaj } \pi'} = \sum_{\pi \in \mathfrak{S}_n} x^{\text{des } \pi} q^{\text{comaj } \pi},$$

so (des, maj) and $(\text{des}, \text{comaj})$ have the same joint distribution on \mathfrak{S}_n . \square

We can also apply Theorem 5.1 to signed permutations and wreath products, making use of Lemma 3.5. Extend maj and comaj to $C_k \wr \mathfrak{S}_n$ by defining for $\pi^\sigma \in C_k \wr \mathfrak{S}_n$:

$$\begin{aligned} \text{maj } \pi^\sigma &= \sum_{j \in \text{Des } \pi^\sigma} j; \\ \text{comaj } \pi^\sigma &= \sum_{j \in \text{Des } \pi^\sigma} (n-j). \end{aligned}$$

The statistic *flag inversion number* is defined for $\pi^\sigma \in C_k \wr \mathfrak{S}_n$ by

$$\text{finv } \pi^\sigma = \text{inv } \pi + \sum_{i=1}^n i \sigma_i.$$

The bijection Θ of Lemma 3.5 maps “Des” on $C_k \wr \mathfrak{S}_n$ to “Asc” on $\mathbf{I}_{n,k} = \mathbf{I}_n^{(k,2k,\dots,nk)}$. In [17, Theorem 3], Θ was shown to have the following additional property.

Lemma 5.5. *The bijection $\Theta : C_k \wr \mathfrak{S}_n \longrightarrow \mathbf{I}_{n,k}$ of Lemma 10 satisfies*

$$\text{finv } \pi^\sigma = \text{inv } \Theta(\pi^\sigma).$$

Theorem 5.6. *For the descent polynomial of $C_k \wr \mathfrak{S}_n$, defined by*

$$W_{n,k}(x) := \sum_{\pi^\sigma \in C_k \wr \mathfrak{S}_n} x^{\text{des } \pi^\sigma},$$

the following q -analogs have all roots real for $q > 0$:

$$W_{n,k}(x, q) = \sum_{\pi^\sigma \in C_k \wr \mathfrak{S}_n} x^{\text{des } \pi^\sigma} q^{\text{finv } \pi^\sigma}, \quad (16)$$

$$W_{n,k}(x, q) = \sum_{\pi^\sigma \in C_k \wr \mathfrak{S}_n} x^{\text{des } \pi^\sigma} q^{\text{comaj } \pi^\sigma}, \quad (17)$$

$$W_{n,k}(x, q) = \sum_{\pi^\sigma \in C_k \wr \mathfrak{S}_n} x^{\text{des } \pi^\sigma} q^{\text{maj } \pi^\sigma}. \quad (18)$$

Proof. By Lemma 5.5, the polynomial (16) is $\mathbf{E}_n^{(k,2k,\dots,nk)}(x, 1, q)$ and the polynomial (17) is $\mathbf{E}_n^{(k,2k,\dots,nk)}(x, q, 1)$, both of which are real-rooted by Theorem 5.1. In contrast to the case for \mathfrak{S}_n , when $k > 1$, the polynomials (17) and (18) are not the same. However,

$$\sum_{\pi^\sigma \in C_k \wr \mathfrak{S}_n} x^{\text{des } \pi^\sigma} q^{\text{maj } \pi^\sigma} = \sum_{\pi^\sigma \in C_k \wr \mathfrak{S}_n} (xq^n)^{\text{des } \pi^\sigma} q^{-\text{comaj } \pi^\sigma} = \mathbf{E}_n^{(k,2k,\dots,nk)}(xq^n, 1/q, 1).$$

By Theorem 5.1, $\mathbf{E}_n^{(k,2k,\dots,nk)}(x, 1/q, 1)$ has all real roots, and therefore $\mathbf{E}_n^{(k,2k,\dots,nk)}(xq^n, 1/q, 1)$ also does, by Remark 2.1. \square

6 The Euler–Mahonian polynomials for \mathfrak{B}_n and $C_k \wr \mathfrak{S}_n$

In the case of \mathfrak{B}_n and $C_k \wr \mathfrak{S}_n$, a different q -analog of the Eulerian polynomial is based on the *flag major index* statistic [1], fmaj , which is defined for $\pi^\sigma \in C_k \wr \mathfrak{S}_n$ by

$$\text{fmaj } \pi^\sigma = k \text{comaj } \pi^\sigma - \sum_{i=1}^n \sigma_i.$$

This definition differs a bit from those appearing elsewhere because of the appearance of comaj , but was it shown in [17] to be equivalent e.g. to [7, 8].

In contrast to comaj and maj from the previous section, fmaj is *Mahonian*, i.e., it has the same distribution as “length” on $C_k \wr \mathfrak{S}_n$ [1]. For that reason, the polynomials

$$F_{n,k}(x, q) = \sum_{\pi \in C_k \wr \mathfrak{S}_n} x^{\text{des } \pi} q^{\text{fmaj } \pi}$$

are referred to as the *Euler–Mahonian polynomials* for $C_k \wr \mathfrak{S}_n$. It was conjectured in [8] that $F_{n,k}(x, q)$ has all roots real for $q > 1$.

In the special case, $k = 2$, of signed permutations, \mathfrak{B}_n , the conjecture was made in [7] that for any $q > 0$, the polynomial $B_n(x, q)$, defined by

$$B_n(x, q) = \sum_{\pi \in \mathfrak{B}_n} x^{\text{des } \pi} q^{\text{fmaj } \pi},$$

has all real roots.

We can now settle these conjectures.

Theorem 6.1. *For the wreath product groups, $C_k \wr \mathfrak{S}_n$, the Euler–Mahonian polynomials*

$$F_{n,k}(x, q) = \sum_{\pi \in C_k \wr \mathfrak{S}_n} x^{\text{des } \pi} q^{\text{fmaj } \pi}$$

have all real roots for $q > 1$.

To prove this theorem, we first relate fmaj to a statistic, fmaj_I, on the inversion sequences $I_{n,k} = \mathbf{I}_n^{(k, 2k, \dots, kn)}$.

For $\mathbf{e} \in I_{n,k}$, define

$$\text{fmaj}_I \mathbf{e} = k \text{ amaj } \mathbf{e} - \sum_{j=1}^n \left\lfloor \frac{e_j}{j} \right\rfloor.$$

It was shown in [17, Theorem 3] that the bijection $\Theta : C_k \wr \mathfrak{S}_n \rightarrow I_{n,k}$ of Lemma 3.5, which maps “Des” on $C_k \wr \mathfrak{S}_n$ to “Asc” on $I_{n,k}$ also satisfies

$$\text{fmaj } \pi^\sigma = \text{fmaj}_I \Theta \pi^\sigma.$$

We thus have

$$F_{n,k}(x, q) := \sum_{\pi \in C_k \wr \mathfrak{S}_n} x^{\text{des } \pi} q^{\text{fmaj } \pi} = \sum_{\mathbf{e} \in I_{n,k}} x^{\text{asc } \mathbf{e}} q^{\text{fmaj}_I \mathbf{e}}. \quad (19)$$

For $n \geq 1$ and $0 \leq i < s_n$, define

$$F_{n,k,i}(x, q) = \sum_{\{\mathbf{e} \in I_{n,k} \mid e_n = i\}} x^{\text{asc } \mathbf{e}} q^{\text{fmaj}_I \mathbf{e}}.$$

Lemma 6.2. *For $n, k \geq 1$ and $0 \leq i < s_n$,*

$$F_{n,k,i}(x, q) = q^{\lfloor i/n \rfloor} \left(\sum_{j=0}^{\ell-1} x q^k F_{n-1,k,j}(x q^k, q) + \sum_{\ell}^{s_{n-1}-1} F_{n-1,k,j}(x q^k, q) \right)$$

where $\ell = \lceil i s_{n-1} / s_n \rceil$, and with initial conditions $F_{1,k,0}(x, q) = 1$ and $F_{1,k,i}(x, q) = x q^{k-i}$ for $i > 0$.

Proof. For $\mathbf{e} = (e_1, \dots, e_n) \in \mathbf{I}_n^{(s)}$ with $e_n = i$, let $j = e_{n-1}$. Then $n-1 \in \text{Asc } \mathbf{e}$ if and only if $j/s_{n-1} < i/s_n$, that is $0 \leq j \leq \ell-1$. Let $T_j = xq^k$ if $j \leq \ell-1$ and otherwise $T_j = 1$. Then

$$\begin{aligned} x^{\text{asc } \mathbf{e}} q^{\text{fmaj } \mathbf{e}} &= q^{-(\lfloor e_1/1 \rfloor + \lfloor e_2/2 \rfloor + \dots + \lfloor e_n/n \rfloor)} \prod_{t \in \text{Asc } \mathbf{e}} (xq^{k(n-t)}) \\ &= q^{-\lfloor i/n \rfloor} T_j q^{-(\lfloor e_1/1 \rfloor + \lfloor e_2/2 \rfloor + \dots + \lfloor e_{n-1}/(n-1) \rfloor)} \prod_{t \in \text{Asc}(e_1, \dots, e_{n-1})} ((xq^k)q^{k(n-1-t)}) \\ &= q^{-\lfloor i/n \rfloor} T_j q^{-(\lfloor e_1/1 \rfloor + \lfloor e_2/2 \rfloor + \dots + \lfloor e_{n-1}/(n-1) \rfloor)} (xq^k)^{\text{asc}(e_1, \dots, e_{n-1})} (q^k)^{\text{amaj}(e_1, \dots, e_{n-1})} \\ &= q^{-\lfloor i/n \rfloor} T_j (xq^k)^{\text{asc}(e_1, \dots, e_{n-1})} q^{\text{fmaj}(e_1, \dots, e_{n-1})}. \end{aligned}$$

For the initial conditions, $0 \in \text{Asc } \mathbf{e}$ if and only if $e_1 > 0$, in which case

$$x^{\text{des } \mathbf{e}} q^{\text{fmaj } \mathbf{e}} = xq^{k \text{ amaj } \mathbf{e} - \lfloor i/1 \rfloor} = xq^{k-i}.$$

□

Proof. of Theorem 6.1

For fixed k , setting $z = 1$, $b_0 = 1$, $b_i = q^{k-i}$, and $m_i = q^{\lfloor i/n \rfloor}$ in Theorem 5.3 gives the recurrence of Lemma 6.2. Thus, by Theorem 5.3, the polynomials $\{F_{n,k,i}(x, q) \mid 0 \leq i < s_n\}$ are pairwise compatible and so, by Lemma 2.2, they are compatible. This implies that the polynomial

$$F_{n,k}(x, q) := \sum_{\pi \in C_k \wr \mathfrak{S}_n} x^{\text{des } \pi} q^{\text{fmaj } \pi} = \sum_{\mathbf{e} \in \mathbf{I}_{n,k}} x^{\text{asc } \mathbf{e}} q^{\text{fmaj } \mathbf{e}} = \sum_{i=0}^{kn-1} F_{n,k,i}(x, q)$$

has all real roots. □

7 Further directions and open questions

In Section 3.2 we showed that the Eulerian polynomial of type B has only real roots as a consequence of Theorem 1.1. The real rootedness was originally proven by Brenti who conjectured that this is part of a larger phenomenon, namely, that the Eulerian polynomials for all Coxeter groups have only real roots. Recently, a preprint appeared on the arXiv [14] which claims to have settled the only missing part of this conjecture, the type D case. A natural question therefore is to ask whether there exists a sequence \mathbf{s} for which the \mathbf{s} -Eulerian polynomial reduces to the type D Eulerian polynomial.

In Sections 5 and 6 we proved that the respective Euler–Mahonian polynomials have only real roots whenever q is nonnegative. The conjectures of both Chow and Gessel [7] and Chow and Mansour [8] have a stronger form that asserts that these polynomials are not only real rooted, but their roots satisfy some separation properties. The stronger form of these conjecture would imply that consecutive polynomials have interlacing roots, that is, they form a Sturm sequence. This would also imply our result that these polynomials have only real roots.

Finally, we note that a corresponding theory of *descent polynomials* for inversion sequences $\mathbf{I}_n^{(s)}$ was developed in [18]. It is likely that similar real-rooted properties hold. This is a topic for future investigation, as is the topic of applications of these descent polynomials.

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